Publisher homepage: www.universepg.com, ISSN: 2707-4625 (Online) & 2707-4617 (Print)

https://doi.org/10.34104/ijmms.021.01010112

International Journal of Material and Mathematical Sciences

Journal homepage: https://www.universepg.com/journal/ijmms

Some Characterizations of the Extended Beta and Gamma Functions: Properties and Applications

Bachioua Lahcene*

Department of Basic Sciences, Prep. Year, P.O. Box 2440, University of Ha'il, Ha'il, Saudi Arabia.

*Correspondence: drbachioua@gmail.com (Dr. Bachioua Lahcene, Associate Professor, Department of Basic Sciences, Prep. Year, P.O. Box 2440, University of Ha'il, Ha'il, Saudi Arabia).

ABSTRACT

The article represents the elementary and general introduction of some characterizations of the extended gamma and beta Functions and its important properties with some various representations. This paper provides reviews of some of the new proposals to extend the form of basic functions and some closed-form representation of more integral functions is described. Some of the relative behaviors of the extended function, the special cases resulting from them when fixing the parameters, the decomposition equation, the integrative representation of the proposed general formula, the correlations related to the proposed formula, the frequency relationships, and the differentiation equation for these basic functions were investigated. We also investigated the asymptotic behavior of some special cases, known formulas, the basic decomposition equation, integral representations, convolutions, recurrence relations, and differentiation formula for these target functions by studying. Applications of these functions have been presented in the evaluation of some reversible Laplace transforms to the complex of definite integrals and the infinite series of related basic functions.

Keywords: Extended, Gamma, Incomplete gamma, Functions, Properties, Applications, and Transformation.

A random variable is expressed in mathematics as a measurable function in a certain period of the probability space adopted in the period (0,1), and generally takes the form of real numerical quantities, where the probability values of a random variable may represent the possible outcomes of a random experiment, or the possible outcomes of a previous experiment. It is in fact, unconfirmed (Ferreira *et al.,* 2000). Probability is a result of many reasons, including that measurements are inaccurate, uncertain, and because of random theoretical results or are produced through incomplete random results tainted in two parts different. The random variable is attributed to a probability distribution that determines the probability of the values that the value of the random variable takes. It can be a discrete random variable on a countable set, connected to a partial period of

INTRODUCTION:

Researchers abstractly consider natural phenomena when modeled according to a classification that includes three cases according to their specific total traceability, the first case as a deterministic value, the second as random, and the third as a chaotic phenomenon whose values are cited in the cases described according to the target models. Stochastic models include the largest aspect of the targeted phenomena. Therefore, the probability and statistics has been the wide field in which researchers go to study random issues, where the random variable is described as a function that expresses the random quantity, a function that expresses the value of the results of a random phenomenon at the resulting values of the function (Blitzstein *et al.,* 2014).

adding additional parameters to the existing distribution to create a new family distribution to represent a combination of random models (Lee *et al.,* 2013).

The probability density function (pdf) of continuous distributions is seen as a function of the random variable X that fulfills the following two conditions:

$$\begin{cases} f_X(x) \geq 0, & for\, all\, values\, of\ x \\ \int_{-\infty}^{+\infty} f_X(x)dx = 1 \end{cases} \quad (1)$$

Through the two conditions mentioned in equation (1) has been many functions that meet these two conditions, and based on the modeling of statistical distributions, many researchers were able to propose functions that took their name. In order to expand any pdf, the following formula was adopted:

$$g_X(x) = \frac{x^k f_X(x)}{E(X^k)} \quad (2)$$

Whereas, $E(X^k) = \int_{-\infty}^{+\infty} t^k f_X(t)dt$ and $g_X(.)$ the new pdf of the random variable X . Many researchers have been able to propose a formula for the pdf as a generalized formula as follows:

$$g_X(x/f, p) = \frac{f_X(x)}{p(F(x))} \quad (3)$$

The formula (3) is defined by Ferreira and Steel, (2006) in order to represent skewed distributions based on symmetrical distribution (Harold *et al.,* 1972). Several important distribution families have been generated and derived, including the T-X family. (Alzatreh *et al.,* 2013; Rudin &Walter, 1987) have studied some of their details. From generalized versions of equations (2) and (3), a more number of continuous distribution families can be created. In this work, we seek to address both extended generalized gamma distribution and extended Beta distribution, and demonstrate some properties and applications (Awadala *et al*., 2020).

History of Gamma Function and its Changes

In order to know most of the new formulas that have been derived from the generalization of the general form of the gamma function and the incomplete functions of the gamma function, it is noted that it is very important to give an integrated review of the history of the emergence and development of integrative functions and how to trade the formulas real numbers, or a combination of both types, which are very rare.

The conditions of the well-known probability density function have been met, and there is still a difference between two random variables with the same probability distribution from the point of view that they are randomly related or independent, and this is clear from the results of choosing the random values of the random variable according to the probability distribution function investigated (Johnson *et al.,* 1994). Since the nineteenth century, many scholars have been developing research on probability distributions that reflect statistical phenomena with a high level of stochastic inference, and allow real phenomena to be traced within appropriate proportions to the real world. Pearson, (1895) was able to present a model for generating continuous distributions emanating from differential equations, which took its name, and the distribution of Pearson models, then the models developed by the researcher Johnson, (1949), followed by Burr models, and other wonderful models that achieved stimulating results, in addition to models That radically changed the view of scientists about probability distributions and their applications (Bachioua, 2013; 2014). Tukey, (1960) was able, through alignment models, to develop methods based on quantitative functions and find a probability distribution that allows the adaptation of continuous and discrete random distributions (Bachioua & Shaker, 2006). With the new developed results achieved by these models and the expansion in the field of application, many articles have appeared interested in trading the probability distributions and searching for the relationship between each other, where I heard the results obtained from making approximate maps of the links between most of the probability distributions and the special case involving each other (AL Jarrah *et al.,* 2014). Interest has expanded in recent years to developing and extending innovative methods for generating statistical distributions, finding some distributions that include most possible models for other distributions that do not conflict with the law of central tendency to achieve general goals and flexibility to reduce margins of errors when genera-ting pseudorandom numbers. Lee *et al.* (2013) noted that most of the methods developed after the 1980s are methods of standardization for several reasons, including that these new methods are based on the idea of combining two existing distributions or

$\exp(-b/t)$ in the form equation (5). We note the following relation equation by (16):

$$\Gamma_b(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{\left(-t-\frac{b}{t}\right)} dt \qquad (6)$$

The formula for the gamma function can be shown in two basic parts by the incomplete upper form of the gamma function and the incomplete lower part of the incomplete gamma function, according to the formula:

$$\Gamma_b(\alpha) = \gamma_b(\alpha, x) + \Gamma_b(\alpha, x)$$

Due to the huge success of the gamma function, many authors have defined and discussed different type of gamma function in recent years. Recently Kobayashi, (1991) considered a generalized gamma function in the generalized form by –

$$\Gamma_r(\alpha, n) = \int_0^{+\infty} t^{\alpha-1} [t+n]^{-r} e^{-t} dt \qquad (7)$$

Kobayashi considered the field of flat-wave diffraction by tape using the Wiener-Hopf technique. Bachioua, (8) has demonstrated a new generalization of the gamma function proposed by Kobayashi earlier by introducing the parameter m the first time and representing the new gamma function through the following formula:

$$\Gamma_r(\alpha, n, m) = \int_0^{+\infty} t^{\alpha-1} [t^m + n]^{-r} e^{-t} dt \qquad (8)$$

Bachioua later defines additional parameters, and studies the convergence of the extended generalized gamma function in terms of six parameters; it used modified forms of the generalized gamma function of the Kobayashi function, which is a special case when Bachioua fixes some parameters of the extended function (8):

$$\Lambda_r(\alpha, n, m, p, \lambda) = \int_0^{+\infty} t^{\alpha-1} [t^m + n]^{-r} e^{-\lambda t^{p-1}} dt \qquad (9)$$

The properties of convergence for the integrative function were studied for the six parameters for α, n, m, p, λ positives and r real numbers (8). Equation (7) is a special case of equation (8), which is itself a special case of equation (9) and can be checked by fixing parameters at special values. The expanded gamma function can be reformulated to more broadly reflect most of the new modifications and add withdrawals to the exponential variables. The proposed formula of seven parameters is given the following formula:

$$\Lambda_r(\alpha, n, m, p, k, \lambda) = \int_0^{+\infty} t^{\alpha-1} [t^m + n]^{-r} e^{-\lambda t^{p-1} + t^{-k}} dt \qquad (10)$$

emanating from the generalized formula, where the gamma function attracted the attention of some of the most prominent scientists Mathematics of all time. Initially the first function was first introduced by Leonhard Euler while pursuing the goal of generalizing the factorization of computations to incorrect values. In a letter dated 08/01/1730 to Christian Gold Bach, Sandifer, (2007) Suggest the first form of the gamma function, as follows:

$$\Gamma(x) = \int_0^1 (-\log(t))^{x-1} dt. \qquad (4)$$

The equation (4) named Euler function; this function belongs to category of the special functions and appears in various area as Zeta Riemann function, Hyper geometric series, and number theory (Abramowitz Milton & Stegun Irene, 1972).Until the middle of the last (twentieth) century, mathematicians relied on hand-made approximation tables in which values were calculated in the case of the gamma function, especially the tables computed by Gauss in 1813 and the ones computed by Legendre in 1825, which were used respectively In calculating the values of changing variables related to the significance of the gamma function values. If let $u = \log(t)$ in equation (1) for $x \succ 0$ we get the function equation in the new form:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt = 2\int_0^{+\infty} t^{2x-1} e^{-t^2} dt \qquad (5)$$

The equation (5) named gamma function and the relation $\Gamma(x+1) = x\Gamma(x)$ is the important functional equation, for integer values the equation becomes $\Gamma(n+1) = n!$.

This led Legendre in 1811 to decompose the gamma function into the incomplete gamma functions, $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ which represent (16): The upper incomplete gamma function is defined as:

$$\Gamma(\alpha, x) = \int_x^{+\infty} t^{\alpha-1} e^{-t} dt$$

Whereas the symbol for the definition of the lower incomplete gamma function is indicated as:

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$$

Because of the importance in simplifying the symbols, the regular form of the gamma function holds, and is defined as;

$$\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x)$$

Choudhury and Zubair were able to extend the scope of these functions by introducing new formula, where when you put a new regularization variable

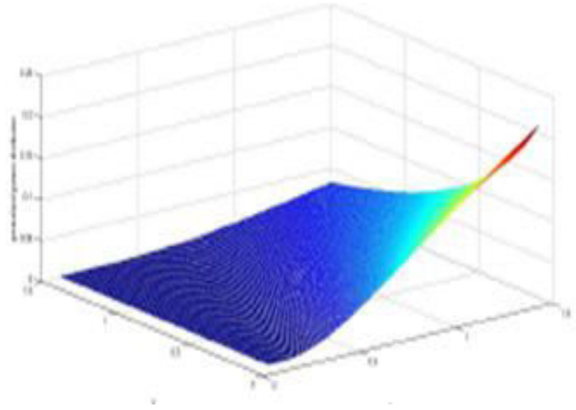

Fig 2: Graph of pdf function λ versus with$r = 0.2; \alpha = 1.5;\ n = 0.5; m = 0.3; p = 0.1;\ b = 1$.

Fig 2, shows the visual representation of the pdf file of the random variable function X, in which the important properties of the proposed model appear, and for the sake of broad investment and maximum benefit from the new properties of the proposed model referred to, the researcher also proposes the extended generalized function of the new incomplete gamma function as the following:

$$\Lambda_r(\alpha,n,m,p,k,\lambda) = \gamma_r(\alpha,n,m,p,k,\lambda : x) + \Lambda_b(\alpha,n,$$

It is time to introduce the proposed new form of the extended incomplete gamma function as:

$$\gamma_r(\alpha,n,m,p,k,\lambda : x) = \int_0^x t^{\alpha-1}\left[t^m + n\right]^{-r} e^{-\lambda\, t^{p-1} - (1-\lambda)\, t^{-k}}\, dt \qquad (11)$$

It is necessary to trace the cases that show the most important additions introduced by the new proposal, it is also useful to search for the relationships between the original beta function and its extension, and in this regard, we first present the relationship between them. With ease and simplicity, the generalization reviewed by researcher Boudjelkha Mohamed, in 2017 can be expressed as (Mohamed *et al.,* 2000) follows:

$$\gamma_0(\alpha,n,1,2,1,0 : x) = \int_0^x t^{\alpha-1} e^{-t-\frac{b}{t}}\, dt \equiv \gamma_b(\alpha,n : x)$$

Proposed Extended BETTA Function

In 1997, Chaudhary *et al.* from a paraphrase, another extension of the classical beta function defined as follows are presented;

$$B_p(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{-\frac{p}{t(1-t)}} dt \quad , p \geq 0$$

In 2018, Mehar *et al.* (Harold & Bertha, 1972) introduced the extended additional extension of the classical beta function defined as follows;

In order to respond to the most important proposed formulas for expanding the formulas of the generalized gamma function by expansion, the researcher seeks to continue to propose a new formula, which is the content of the later part of this article, as it is determined according to the parameters domain that makes the defined integral in terms of eight convergent parameters. The probability density function diagrams (pdf) for X indicate different values for the parameters of the gamma function for the most important changes in the shape of the graphics, which allows identifying the areas of possible applications, which are shown in the **Fig 1**.

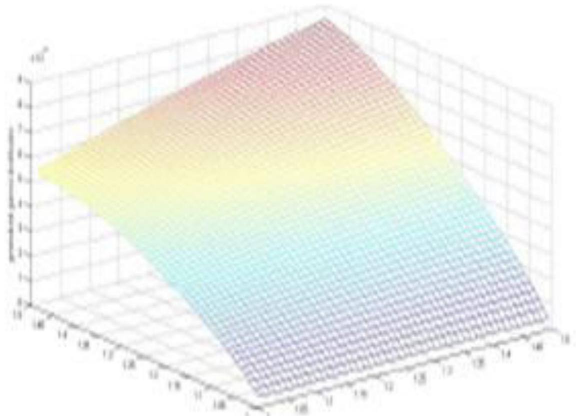

Fig 1: Graph of pdf function α versus with$r = 1.3; \lambda = 1.2;\ n = 0.9; m = 1; p = 0.2;\ b = 0.7$.

In **Fig 1**, we provide the visual representation of the pdf of the X, which shows the important mathematical properties of the proposed generalized model formula.

Proposed extended GAMMA function:

In this section and after reviewing the theoretical literature through the contents of the articles whose content has been examined and the most important modifications proposed by many researchers and authors on the gamma function, it becomes clear that the need for research and application requires reformulating the extended gamma function formula, and through the proposed model the researcher presents the new formula As follows (6):

$$\Lambda_r(\alpha,n,m,p,\lambda,k) = \int_0^{+\infty} t^{\alpha-1}\left[t^m + n\right]^{-r} e^{-\lambda\, t^{p-1} - (1-\lambda) t^{-k}}\, dt \qquad (11)$$

In order to understand the most important ideas, the pdf function of the random variable X has been drawn for different selected values of some parameters included in the function, which are from the **Fig 2**.

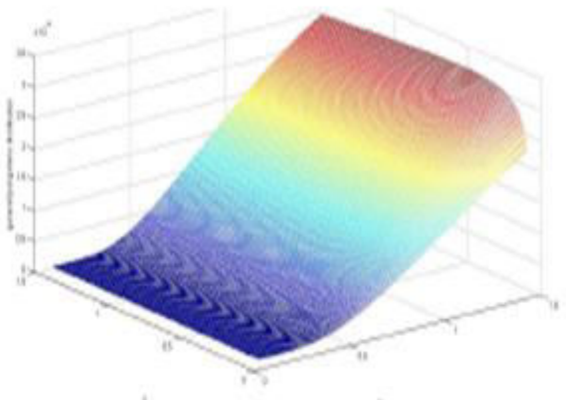

Fig 3: Graph of pdf function m versus with $r = 0.3; \alpha = 1.01;\ n = 0.4; \lambda = 0.7; p = 0.01;\ b = 1.4$.

It's time to ask the most important question about the possibility of finding a more comprehensive formula or additional specific conditions for the transformation function that preserve the convergence of the integral of the beta transformation function. With this, we introduce a new extension of the classic beta function as follows:

$$B_p(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-pt(1-t)}dt \quad ,|p|\prec\infty$$

We recall that the transformation of the beta function (Mehar *et al.*, 2018) of $f(t)$ can be redefined as:

$$BT(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t)dt$$

From the first suggestion, note the formula of the function $f(t)=e^{-\frac{p}{t(1-t)}}$, and in the second suggestion, note that the formula of the function $f(t)=e^{-pt(1-t)}$. In order to show some important cases, some cases of the pdf function of the random variable function X are represented for some different selected values of the function parameters, which are shown in the **Fig 3**. In **Fig 3**, the graph of the pdf function of the random variable X is presented, which shows some important properties of the proposed model.

$$B_\lambda(\alpha,\beta,k,p)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left\{-\lambda\, t^k(1-t)^p\right\}dt, \alpha,\beta,p,k,\lambda\in IR^+$$

the random variable X for some different selected values of the shape parameters (4).

In **Fig 4**, the graph of the pdf function of the random variable X is presented, which shows some important properties of the previously mentioned model.

Property 1: The formula for the function $B_\lambda(\alpha,\beta,k,p)$ can be re-expressed in the following simple form:

$$B_\lambda(\alpha,\beta,k,p)=\int_0^{+\infty} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\exp\left\{-\lambda\, x^k(1+x)^{-(k+p)}\right\}dx$$

Proof: First, we change the unknown $x=t\,(1-t)^{-1}$, this confirms that when t he changes $0\ to\ 1$, then x he changes from $0\ to\ +\infty$ and $dt=(1-x)^{-2}dx$

Mathematical Properties and Recurrence Relation

First, we try to find similarities between the two previously proposed models, and then we find the relationship between the two models of the beta function and the proposed gamma function model. It is also simply possible to ensure that the proposed integral in the above-mentioned formula is a convergent and positive value. A set of references can be used to assert in cases of gamma function formula in terms of six parameters (6). The study of this generalized case gives us an idea of the mechanism of finding special cases that express the totality of the previous cases. In order to clarify some of these ideas, we review some forms of the pdf function of

$$\begin{aligned}B_\lambda(\alpha,\beta,k,p)&=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left\{-\lambda\, t^k(1-t)^p\right\}dt\\ &=\int_0^{+\infty}\left(\frac{x}{1+x}\right)^{\alpha-1}\left(1-\frac{x}{1+x}\right)^{\beta-1}\exp\left\{-\lambda\left(\frac{x}{1+x}\right)^k\left(1-\frac{x}{1+x}\right)^p\right\}(1-x)^{-2}dx\\ &=\int_0^{+\infty} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\exp\left\{-\lambda\, x^k(1+x)^{-(k+p)}\right\}dx\end{aligned}$$

Property 2: The formula for the function $\Lambda_r(\alpha,n,m,p,\lambda,k)$ can be re-expressed in the following simple form:

$$\Lambda_r(\alpha,n,m,p,\lambda,k)=\int_0^1 x^{\alpha-1}(1+x)^{-(\alpha+1)}\left(\left(\frac{x}{1+x}\right)^m+n\right)^{-r}\exp\left\{-\lambda\left(\frac{x}{1+x}\right)^{p-1}-(1-\lambda)\left(\frac{x}{1+x}\right)^{-k}\right\}dx$$

Proof: First, we change the unknown $x = t\,(1-t)^{-1}$, this confirms that when t he changes $0\ to +\infty$, then x he changes from $0\ to\ 1$ and $dt = (1-x)^{-2}\,dx$

$$\Lambda_r(\alpha,n,m,p,\lambda,k) = \int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda\, t^{p-1}-(1-\lambda)t^{-k}}\,dt$$

$$= \int_0^{+1}\left(\frac{x}{1+x}\right)^{\alpha-1}\left(\left(\frac{x}{1+x}\right)^m+n\right)^{-r}\exp\left\{-\lambda\left(\frac{x}{1+x}\right)^{p-1}-(1-\lambda)\left(\frac{x}{1+x}\right)^{-k}\right\}(1-x)^{-2}\,dx$$

$$= \int_0^{1} x^{\alpha-1}(1+x)^{-(\alpha+1)}\left(\left(\frac{x}{1+x}\right)^m+n\right)^{-r}\exp\left\{-\lambda\left(\frac{x}{1+x}\right)^{p-1}-(1-\lambda)\left(\frac{x}{1+x}\right)^{-k}\right\}dx$$

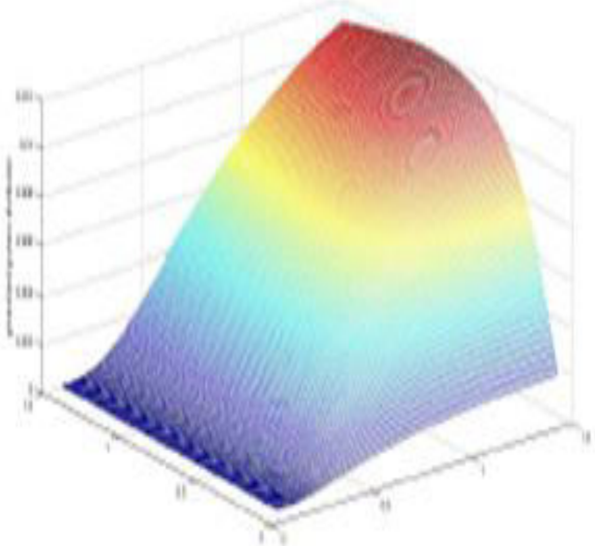

Fig 4: Graph of pdf function n versus with $r = 1; \alpha = 0.3;\ m = 1.2; \lambda = 0.7; p = 0.6;\ b = 1$.

Remark: By examining both property (1) and property (2), the special cases of both proposed functions can be easily expressed, and thus there is still a wide common field between the two functions, which expands the scope of participation and inclusiveness of the gamma function with respect to the beta function, which has been expanded, and from During the presented proposal, we review some forms of the pdf function of the random variable X for different selected values of the function parameters as in the **Fig 5**.

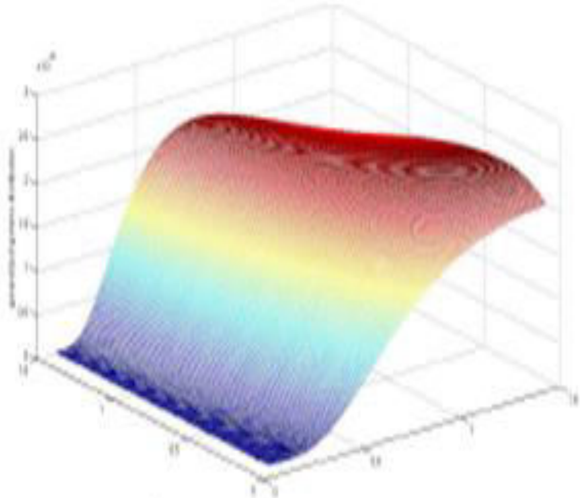

Fig 5: Graph of pdf function p versus with $r = 1.4; \alpha = 1.2;\ m = 1.3; \lambda = 1; n = 0.1;\ b = 1.5$.

In **Fig 5**, the graphical representation of some cases of the pdf function of the random variable X is presented, which shows the important properties of the proposed model.

Convergence of Extended BETA and GAMMA Function

In this section, we will talk about some special cases of beta and gamma extended functions that fall into the category of these convergent extended integrals. Therefore, we will discuss again and mainly the affinity of this particular function of the proposed functions, which is an improper integral.

Convergence of the Extended BETA Function

$B_\lambda(\alpha,\beta,k,p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left\{-\lambda\, t^k(1-t)^p\right\}dt$ Case (a): for $\lambda = 0$

Case (1): for $\alpha \geq 1, \beta \geq 1$ the considered integral has a definite value, and who it is a convergent integration.

Case (2): for $0 \prec \alpha \prec 1, 0 \prec \beta \prec 1$, let $0 \prec c \prec 1$, then

$$B_0(\alpha,\beta,k,p)\equiv B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\int_0^c t^{\alpha-1}(1-t)^{\beta-1}dt+\int_c^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

Consider $I_1=\int_0^c t^{\alpha-1}(1-t)^{\beta-1}dt$, let $f(t)=t^{\alpha-1}(1-t)^{\beta-1}$ and $g(t)=t^{\alpha-1}$ then

$\lim\limits_{t\mapsto 0} f(t)/g(t)=\lim\limits_{t\mapsto 0} t^{1-\alpha}t^{\alpha-1}.(1-t)^{\beta-1}=\lim\limits_{t\mapsto 0}=(1-t)^{\beta-1}=1$, and now for the comparison we have to take another function $\int_0^c \frac{dt}{t^{-\alpha+1}}$ converge for $1-\alpha\prec 1$, and then I_1 converge for $\alpha\succ 0$.

Consider $I_2=\int_c^1 t^{\alpha-1}(1-t)^{\beta-1}dt$, let $f(t)=t^{\alpha-1}(1-t)^{\beta-1}$ and $g(t)=(1-t)^{\beta-1}$, then

$\lim\limits_{t\mapsto 1} f(t)/g(t)=(1-t)^{-\beta+1}t^{\alpha-1}.(1-t)^{\beta-1}=\lim\limits_{t\mapsto 1}=t^{\alpha-1}=1$, and now for the comparison we have to take another function $\int_c^1 \frac{dt}{(1-t)^{-\beta+1}}$ converge for $1-\beta\prec 1$, and then I_2 converge for $\beta\succ 0$. We can conclude from the behavior of this improper integral this test integral. Therefore, this converges for $0\prec\alpha\prec 1, 0\prec\beta\prec 1$.

The improper integral $B_\lambda(\alpha,\beta,k,p)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\{-\lambda\, t^k(1-t)^p\}dt$ converge for $\lambda=0,\ \alpha\succ 0, \beta\succ 0, k,p\in IR$.

Case (b): for $\lambda\succ 0$

The inequality - $0\le\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\{-\lambda\, t^k(1-t)^p\}dt\le\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt\equiv B(\alpha,\beta)$

Satisfy the improper integral converges for $\alpha\succ 0, \beta\succ 0, \lambda\ge 0, p,k\in IR$.

Case(c): for $\lambda\prec 0$, then $-\lambda\, t^k(1-t)^p\ge 0$ for $p,k\in IR$.

The real exponential function can be described in several equivalent ways, which is the most common way, as it is defined by the power series as follows (Rudin & Walter, 1987).

$$\exp\{-\lambda\, t^k(1-t)^p\}=\sum_{i=0}^{+\infty}\frac{\left(-\lambda\, t^k(1-t)^p\right)^i}{(i)!}=\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}t^{ik}(1-t)^{ip}$$

$$B_\lambda(\alpha,\beta,k,p)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\{-\lambda\, t^k(1-t)^p\}dt=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}t^{ik}(1-t)^{ip}dt$$

$$=\int_0^1\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}t^{ik}(1-t)^{ip}t^{\alpha-1}(1-t)^{\beta-1}dt=\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}\int_0^1 t^{\alpha+ik-1}(1-t)^{\beta+ip-1}dt$$

$$=\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}B(\alpha+ik,\beta+ip)$$

The beta function $B(\alpha+ik,\beta+ip)$ defend for all $\alpha+ik\succ 0, \beta+ip\succ 0$ where $i=0,1,2,\ldots,+\infty$,and the series $\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}=\exp\{-\lambda\}$ converge, then the bounded of series

$$0\le\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}B(\alpha+ik,\beta+ip)\le Max_{i=0}^{+\infty}[B(\alpha+ik,\beta+ip)]\sum_{i=0}^{+\infty}\frac{(-\lambda)^i}{(i)!}$$

The Function converges for $\alpha\succ 0, \beta\succ 0, \lambda, k, p\in IR$.

$$B_\lambda(\alpha,\beta,k,p)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\{-\lambda\, t^k(1-t)^p\}dt$$

Convergence of the extended GAMMA function:

$$\Lambda_r(\alpha,n,m,p,\lambda,k)=\int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt$$

Case (a): for $r=0$, for $\lambda=1, p=2$ the integral is represent gamma integral defined by:

$$\Lambda_0(\alpha,n,m,2,1,k)=\int_0^{+\infty} t^{\alpha-1}e^{-t}\,dt\equiv\Gamma(\alpha)$$

Case (1): for $\alpha\geq 1$, the integral is bounded in and for $0\leq t\leq a$, where a is arbitrary we check the convergence of $\int_c^{+\infty} t^{\alpha-1}e^{-t}dt$. Consider $f(t)=t^{\alpha-1}e^{-t}$ and $g(t)=t^{-\alpha}, \alpha\succ 1$, then by comparison test integral, the $\lim\limits_{t\mapsto+\infty}\dfrac{f(t)}{g(t)}=\lim\limits_{t\mapsto+\infty} t^{2\alpha-1}e^{-t}=0$. Then $\Gamma(\alpha)$ converges for $\alpha\geq 1$.

Case (2): for $0\prec\alpha\prec 1$, then $\int_0^{+\infty} t^{\alpha-1}e^{-t}dt=\int_0^{c} t^{\alpha-1}e^{-t}dt+\int_c^{+\infty} t^{\alpha-1}e^{-t}dt$, the integral $\int_c^{+\infty} t^{\alpha-1}e^{-t}dt$ converge. Suppose $f(t)=t^{\alpha-1}e^{-t}$ and $g(t)=t^{\alpha-1}, \alpha\succ 1$, then by comparison of integral test, the $\lim\limits_{t\mapsto 0}\dfrac{f(t)}{g(t)}=\lim\limits_{t\mapsto 0} e^{-t}=e^{-0}=1\neq 0$, then $\int_0^{c} t^{\alpha-1}dt$ converges for $0\prec\alpha\prec 1$.

Then $\Gamma(\alpha)$ converges for $0\prec\alpha\prec 1$. Case (1) and Case (2) prove the convergence of gamma function

$$\Gamma(\alpha)=\int_0^{+\infty} t^{\alpha-1}e^{-t}dt, \alpha\succ 0.$$

Case (3): for $\alpha\leq 0$, then $\int_0^{+\infty} t^{\alpha-1}e^{-t}dt=\int_0^{c} t^{\alpha-1}e^{-t}dt+\int_c^{+\infty} t^{\alpha-1}e^{-t}dt$, the integral $\int_c^{+\infty} t^{\alpha-1}e^{-t}dt$ converge. Suppose $f(t)=t^{\alpha-1}e^{-t}$ and $g(t)=t^{\alpha-1}, \alpha\succ 1$, then by comparison integral test, the $\lim\limits_{t\mapsto 0}\dfrac{f(t)}{g(t)}=\lim\limits_{t\mapsto 0} e^{-t}=e^{-0}=1\neq 0$, then $\int_0^{c} t^{\alpha-1}dt$ diverges for $\alpha\leq 0$. Then $\Gamma(\alpha)$ diverges for $\alpha\leq 0$. Hence, for Case (1), Case (2) and Case (3) prove the convergence of gamma function $\Gamma(\alpha)=\int_0^{+\infty} t^{\alpha-1}e^{-t}dt, \alpha\succ 0$. The improper integral $\Gamma_\lambda(\alpha)=\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt$ is converges.

Case (b): for $r=0$, for $\lambda=0, p=2$ the integral is represent gamma integral defined by:

$\Lambda_0(\alpha,n,m,p,0,k=-2)=\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t}dt\equiv\Gamma(\alpha)$, similar result to case (a).

Case (c): for $r\succ 0$, for $t^m\leq t^m+n$ then $t^{-rm}\geq\left[t^m+n\right]^{-r}$. The inequality

$$0\leq\int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt\leq\int_0^{+\infty} t^{\alpha-rm-1}e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt\equiv\Gamma_\lambda(\alpha)$$

Satisfy the improper integral converges for $\alpha\succ 0, \beta\succ 0, m\prec\dfrac{\alpha}{r}, n\geq 0, \lambda, p, k\in IR$.

Case (d): for $r\prec 0$, then for $0\leq t^{\alpha-1}\left[t^m+n\right]^{-r}\leq t^{\alpha-1}\left[t^m+n\right]^{-[r]+1}$ and let $s=[r]+1\in IN$

$0\leq t^{\alpha-1}\left[t^m+n\right]^{-r}\leq t^{\alpha-1}\left[t^m+n\right]^{-[r]+1}=t^{\alpha-1}\left[t^m+n\right]^{s}=\sum\limits_{i=1}^{S}\begin{pmatrix} i\\ s\end{pmatrix}(n)^{s-i}t^{\alpha+mi-1}$, then inequality

$$0\leq\int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt\leq\sum_{i=1}^{S}\begin{pmatrix} i\\ s\end{pmatrix}(n)^{s-i}\int_0^{+\infty} t^{\alpha-rm-1}e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt$$

, and the;

$$0\leq\int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt\leq\sum_{i=1}^{S}\begin{pmatrix} i\\ s\end{pmatrix}(n)^{s-i}\int_0^{+\infty} t^{\alpha-m\,i-1}e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt\leq\sum_{i=1}^{S}\begin{pmatrix} i\\ s\end{pmatrix}(n)^{s-i}\Gamma(\alpha-m\,i)$$

and the $\Gamma(\alpha - m\,i)$ converge to $(\alpha - m\,i) \succ 0$, for all i then $\alpha - m\,r \succ 0 \Rightarrow m \prec \dfrac{\alpha}{r}$ and for $r \to -\infty, \dfrac{\alpha}{r} \to 0, \alpha \succ 0.$ and the integral of extended gamma function

$$\Lambda_r(\alpha, n, m, p, \lambda, k) = \int_0^{+\infty} t^{\alpha-1}\left[t^m + n\right]^{-r} e^{-\lambda t^{p-1} - (1-\lambda)t^{-k}}\, dt \text{ is converge for } m \prec 0.$$

1991; Kabish, 1991; Bondesson, 1992) which provided many formulas of interest deep extended in engineering applications and reliability. Many researchers continued to study many special cases of the distribution that fall within the gamma distributions, and to show some special basic applications of paramount importance, including Agarwal and Al-Salah, (2001) (3) when studying risk and reliability rates. Researcher Bachioua, (2004) was able to propose a general extension of the gamma distribution in terms of six basic parameters. (8). In recent years, several studies have emerged that have focused a new hybrid methodology called "distribution generators" on creating extended and extended formulas, generating greater flexibility in the target distribution target. These generators are often used when constructing these current distributions in terms of one or several parameters, and with previously known structural properties. In order to show the most important applied cases, some forms of the pdf function were studied for some special parameters of the random variable X for some different selected values of the parameters of the target distribution, which are shown in the **Fig 6**.

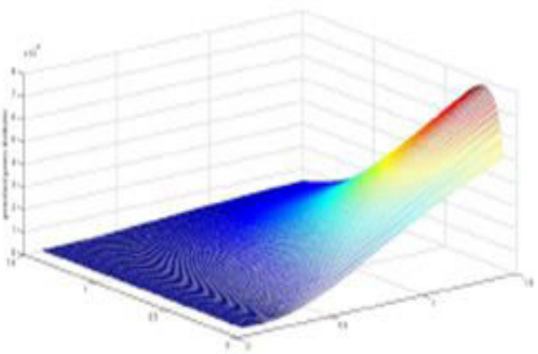

Fig 6: graph of pdf function r versus with $p = 0.01; \alpha = 1.01;\ m = 0.3; \lambda = 1.3; n = 0.4;\ b = 0.3$.

In **Fig 6**, a representation of the pdf function of the random variable X is presented, from which some important properties of the proposed extended model. Enable de Pascoa *et al.* (2011) from the study of some generalized cases of the gamma distribution in its own cases, and through it began to consider the extension of the gamma distribution family resulting

Proposed Extended GAMMA Model Distribution

In this section, we look at the background of the gamma distribution and the most important recent extensions to it. The distribution of the gamma arose from the random variable, according to the authors Johnson *et al.* (1994) in their book that dealt with the subject thoroughly, and that the first to suggest the distribution was Laplace, (1836) when studying some applied cases including the later variable "constant precision" (19), it was used in several areas including the distribution of Gamma for waiting time data modeling, and data systems reliability study. References specializing in probability distributions indicate that Amoroso, (1925) made the first generalization of the gamma distribution when he examined the generalization on the application of income rates and the financial issues of the distribution by representing the model in terms of three parameters. During the twenty years of the last century, some researchers were able to make a first generalization of the distribution presented by Stacey (1962), where he was able to add the location parameter, and then continued studies and research competent in studying the characteristics of the distribution and its basic applications, researchers Mudholkar and Srivastava, (1993) referred to the method Exponential to derive distribution models in its own cases that Stacy previously generalized. In particular, statisticians seek to use the method of deep analysis of the data used, but it becomes difficult when dealing with some intertwined, complex phenomena using classical statistical methods based on the method of non-generalized distributions. These new cases studied and studied by specialized statisticians, motivated them to seek to develop rhythm and comprehensive representation, whether in terms of continuous distributions, or models capable of expressing more clearly, and studying the special properties of the target phenomena data. Some researchers have been able to provide a series of other generalizations complementing the previous ones, such as (Bradley, 1988; Klebaner, 1989; Bondesson, 1989; Lee & Gross, which shows the important properties of the proposed model. For a review of methods for constructing continuous distributions of the T-R{Y} family, Lee *et al.* (2013), where several attempts to study different distributions in the T-R{Y} family have been observed in the past (Rudin & Walter, 1987) and variants generating the generalized odd gamma distribution (Hosseini *et al.,* 2018). Note by Hosseini *et al.* (2018) that the adoption of a mixture of pseudorandom number-generating models of the gamma distribution with a generalized transformation, as it is known to be able to generate highly flexible distributions, has significant advantages in data analysis (adjustment control and provides more flexibility compared to the baseline distribution adopted in quasi-number generation). Randomization by alignment, and produces a slight skew for symmetric distributions, and other desirable properties in the simulation (Hosseini *et al.,* 2018). Given the recent results achieved during the previous studies, the researcher believes that it is necessary to find a general formula to expand the range of previous distributions, and provide researchers with more flexibility to represent continuous data. The researcher proposes the extended formula for the specific probability density function for the random variable of this distribution, as follows:

from the extended beta distribution, which was investigated by Al-Zatira *et al.* (2013) of the TR (W) family, and among the pseudorandom number generators most used in previous years, has been called a gamma generator see (Zografos & Balakrishnan, 2009; Ristıc & Balakrishnan, 2012; Tobari & Montazari, 2012), the logarithmic gamma generator see Aminial, (2014), and the experimental Weibull extended generator of the Weibull generator (Alzaatreh *et al.,* 2013) and Bourguignon *et al.* (2014). Some figures for the pdf function of the random variable X have been plotted for some different selected values of the model parameters, which are shown in the **Fig 7**.

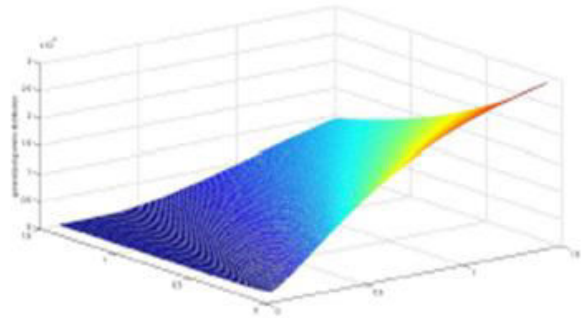

Fig 7: Graph of pdf function b versus with $p = 0.6; \alpha = 1.5;\ m = 1.2; \lambda = 1.4; n = 1.5;\ r = 1$.

In **Fig 7**, the graphical representation of the pdf function of the random variable X is presented,

$$f(\alpha,n,m,p,\lambda,k,r;x) = \frac{x^{\alpha-1}}{\Lambda_r(\alpha,n,m,p,\lambda,k)}\left[x^m+n\right]^{-r}\exp\left\{\lambda x^{p-1}+(1-\lambda)x^{-k}\right\}$$

interval $[0,1]$ with two positive forms parameters, which are indicated by α , and β, which are indicated by the basic formula, which appears as a function of the random variable, α and β is indicated by the form of the probability density distribution function, and control distribution form.

$$B(\alpha,\beta;x) = \frac{1}{B(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

One of the most important applications of the β distribution is used to represent the uncertainty in the probability of a random event, as it is also used to describe empirical data and predict the random behavior of percentages and fractions of pseu-dorandom values, where the range of the results obtained, usually known, becomes between zero and one.The β distribution is usually applied to model some behavior of stochastic variables that is limited to setting values α and β in intervals of limited length,

Where,

$$\Lambda_r(\alpha,n,m,p,\lambda,k) = \int_0^{+\infty} t^{\alpha-1}\left[t^m+n\right]^{-r} e^{-\lambda t^{p-1}-(1-\lambda)t^{-k}}\,dt$$

, $\alpha,n,m,p,\lambda,k,r \in IR$.

Proposed Extended BETA Model Distribution:

The references can be examined and it is easy to find that the beta function was first used by the researcher Wallis, (1655), then the researcher Euler, (1730) when studying the properties of this gamma function, both J. used many of the properties of this function and its applications, Binet, (1839) used the name "beta function", and in the period between 1877 and 1888 the beta distribution was studied by several researchers, who are referred to in the same article by (Bigler, 1888; Schonholzer, 1877) Harold Jeffreys & Bertha Jeffreys, (1972). Many researchers in probability theory and statistics indicate that the beta distribution is a family of continuous probability distributions that are defined across the time

CONCLUSION:

The proposed models allow providing great opportunities for modeling operations of random phenomena, in addition to providing experiments based on pseudo-random numbers generated using simulation and skew methods, and this gives a wider scope for the applications of these functions in proposing comprehensive probability distributions, in addition to physical applications related to generalized functions.

CONFLICTS OF INTEREST:

The author declares that this article content has no conflict of interest.

REFERENCES:

1) Abramowitz Milton, Stegun Irene, A. (1972). "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", National Bureau of Standards Applied Mathematics Series, Tenth Printing, with corrections,
2) Agarwal SK, Kalla SL. (1996). "Generalized Gamma Distribution and its Application in Reliability", *Communication in Statistics: Theory and Method*, **25**(1), pp. 201-210, https://doi.10.1080/03610929608831688
3) AL jarrah M A, Lee Carl, Famoye Felix, (2014). "On Generating T-X Family of Distributions Using Quantile Functions", *J. of Statistical Distributions and Applications,* pp. 1-17. https://doi.org/10.1186/2195-5832-1-2
4) Alzaatreh, A, Lee, C, Famoye, F. (2013). "A New Method for Generating Families of Continuous Distributions", *Metron*, **71**(1), pp.63-79, https://doi.org/10.1007/s40300-013-0007-y
5) Arslan Nasir, Farrukh Jamal, Christophe Chesneau, Muhammad Hussain Tahir, (2019). The Odd Generalized Gamma-G family of Distributions: Properties, Regressions and Applications, HAL archive-ouvertes, pp.1-26, https://doi.org/10.6092/issn.1973-2201/8665
6) Awadala AS, Elfaky AE, Marouf AAS. (2020). Influence of high power Nd: YAG laser on hardness and surface properties of zirconium silicate, *Int. J. Mat. Math. Sci*., **2**(3), pp. 39-44. https://doi.org/10.34104/ijmms.020.039044
7) Bachioua L., and Shaker, H. S. (2006)."On Extended Tukey Lambda Distribution Models", *Jordan Journal of Applied Science Natural Sciences*, **8**(2), pp.25-32.

and in a wide variety of disciplines where pseudo-random counter generation is needed. As a reminder, the value of the β distribution lies in a variety of smooth shapes intended to represent some cases that can assume that there is diversity in the two parameters, α and β, where if the parameters are equal we get an important special case, and the distribution is symmetric. If either parameter is 1 or the other parameter is greater than 1, the distribution will be J-shaped. If the value of α is greater than β the distribution will be negatively skewed (most values α and β are close to the extreme value), cases that allow more random phenomena to be represented.

DISCUSSION:

The importance of the proposal is to extend the applications of the extended beta and gamma function descriptions in extending the usage forms for each of the transformation functions for both the gamma function and the beta function, and by adding weights that result from the additional parameters, and new forms that allow the creation of general and complex integrals that allow the development of specific tasks for the teachers. These extensions add important applications in the use of probability distributions, tracking functions with high efficiency and reliability, and during the study of the range of guesses and predictions for extended problems that express mixed phenomena, which is an important part of the proposed model. The proposal for the extension of the gamma and beta functions, and the incomplete functions for each of them, also provides for some requirements resulting from the extension of the previous results for each of the gamma and gamma incomplete functions in a natural and simple way that allows achieving the target results, which are often difficult to produce in a simple form. Of course, some results similar to the previous results may have any extension of the extended functions indicated in the formula. What we asked for and got is that the extension results should be no less elegant or more complex than the original function of the canonical gamma, beta functions, since the numerical values of $\Lambda_r(\alpha,n,m,p,\lambda,b)$ and $\gamma_r(\alpha,n,m,p,\lambda,b:x)$ can easily be obtained by most mathematical software of most mathematical programs can easily be obtained by alignment and simulation.

Fractional Calculus and Applied Analysis, **4** (3), pp.303-326.

18) Ferreira, JTAS, Steel, MFJ, (2000). A Constructive Representation of Univariate Skewed Distributions", *J Am Stat Assoc,* **101**, pp.823–829. https://doi.org/10.2307/27590738
19) Harold Jeffreys, Bertha Jeffreys, (1972). "Methods of Mathematical Physics", third edition, *Cambridge University Press*, Zbl 0238.00004.
20) Hosseini, B., Afshari, M. and Alizadeh, M. (2018). "The Generalized Odd Gamma-G Family of Distributions: Properties and Applications", *Austrian Journal of Statistics,* **47**, pp.69-89. https://doi.org/10.17713/ajs.v47i2.580
21) Johnson, NL, Kotz, S, Balakrishnan, N. (1994)."Continuous Univariate Distributions", **1**, 2nd ed. *John Wiley & Sons*, Inc., New York.
22) Kobayashi, K. (1991). "On a Generalized Gamma Function Occurring in Diffraction Theory", *Journal of Physical Society of Japan*, **60**, pp.1501-1512. https://doi.org/10.1143/JPSJ.60.1501
23) Lee, C, Famoye, F, Alzaatreh, A. (2013). **"**Methods for Generating Families of Univariate Continuous Distributions in the Recent Decades", *WIREs Computational Statistics*, **5**, pp. 219–238. https://dergipark.org.tr/en/pub/hujms/issue/45735/577353
24) Mehar Chand, Hanaa Hachimi, Rekha Rani, (2018). "New Extension of Beta Function and Its Applications", *International Journal of Mathematics and Mathematical Sciences*, **25**, Article ID 6451592, pp. 25. https://doi.org/10.1155/2018/6451592
25) Rudin, Walter, (1987). "Real and Complex Analysis", (3rd Ed.). *New York: McGraw-Hill.*
26) Sandifer, C. Edward, (2007). "The Early Mathematics of Leonhard Euler", Mathematical Association of America, Washington, DC.
27) Sneddon. I. N. (1979). "The Use of Integral Transforms", *Tata McGraw-Hill*, India.
28) Steigerwald, Douglas G. (2019). "Introduction to Measure Theory", University of California, Santa Barbara. Retrieved September, **30**.

8) Bachioua, L. (2004). "On Extended and Reliability General Mixture Gamma Distribution Model", *A Dissertation Submitted to The* College of Science, University of Baghdad in Partial Fulfillment of the Requirements for The Degree of Doctor of Philosophy (PhD) of Science in Mathematics, University of Baghdad, Iraq.
9) Bachioua, L. (2006). "On Generalized Gamma Distribution Function", First conference in Mathematics, Department of mathematics, college of Applied science and Mathematics Zarqa Private University Amman; 18-20 April, pp. 406-416. Jordan.
10) Bachioua, L. (2013). "On Pearson Families of Distributions and its Applications", *African Journal of Mathematics and Computer Science Research*, pp. 108-117.
11) Bachioua L. (2013). "Extended Generalized Gamma Function and Same its Applications", *Intl. Arch. Appl. Sci. Tech*., **4**(3), pp.16-30. https://www.soeagra.com/iaast/iaastsept2013/3.pdf
12) Bachioua, L. (2014). "On Extended Burr System Family Distribution with Their Application in Quality", *J. of ISOSS*, **1**(1), pp.31-46.
13) Blitzstein, Joe, Hwang, Jessica, (2014). "Introduction to Probability", *CRC Press.*
14) Boudjelkha Mohamed T., Chaudhry. M. Aslam, (2000). "On the Approximation of a Generalized Incomplete Gamma Function Arising in Heat Conduction Problems", *J. of Mathem. Anal., and Appl.*, **248**, pp.509-519. https://core.ac.uk/download/pdf/82795845.pdf
15) Bourguignon, M., Silva, R.B., Cordeiro, G.M. (2014). "The Weibull- G family of probability distributions", *J. of Data Sci.*, **12**, 1253-1268, https://www.jds-online.com/files/JDS-1210.pdf
16) Chaudhry M. A., Qadira A., Rafique M., and Zubair S. M. (1997)."Extension of Euler's Beta Function", *J. of Computational and Applied Mathematics*, **78** (1), pp.19–32. https://doi.org/10.1016/S0377-0427(96)00102-1
17) Chaudhry M. Aslam, and Zubair S.M. (2001). "Extended Gamma and Digamma Functions",

Citation: Lahcene B. (2021). Some characterizations of the extended beta and gamma functions: properties and applications, *Int. J. Mat. Math. Sci.*, **3**(5), 101-112. https://doi.org/10.34104/ijmms.021.01010112